

GUTPA/01/08/01

DCPT-01/63

Quantum Numbers of AGUT Higgs Fields from the Quark-Lepton Spectrum

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Abstract

A series of Higgs field quantum numbers in the anti-grand unification model, based on the gauge group $SMG^3 \times U(1)_f$, is tested against the spectrum of quark and lepton masses and mixing angles. A more precise formulation of the statement that the couplings are assumed of order unity is given. It is found that the corrections coming from this more precise assumption do not contain factors of order of the number of colours, $N_c = 3$, as one could have feared. We also include a combinatorial correction factor, taking account of the distinct internal orderings within the chain Feynman diagrams in our statistical estimates.

Strictly speaking our model predicts that the uncertainty in its predictions and thus the accuracy of our fits should be $\pm 60\%$. Many of the best fitting quantum numbers give a higher accuracy fit to the masses and mixing angles, although within the expected fluctuations in a χ^2 . This means that our fit is as good as it can possibly be.

1 Introduction

Almost the only window of information available to glimpse the laws of nature beyond the Standard Model is at present provided by the ca 20 parameters, mainly already measured but not predicted by the Standard Model itself. Among these, the 13 parameters comprising the masses, mixing angles and CP violating phase for the fermions—quarks and leptons—make up more than half. We have earlier put forward a model [1] as a candidate for explaining the orders of magnitudes of the masses for the six quarks and the three charged leptons. In this anti-grand unification theory (AGUT) the smallness of most of the Standard Model Yukawa couplings is to be understood as a consequence of there existing some approximately conserved gauge charges forbidding these Yukawa couplings [2]. We actually managed in our AGUT model to successfully fit all the 9 charged particle masses¹ and the three mixing angles using only three Higgs field vacuum expectation values, in addition to the usual Weinberg Salam Higgs field.

In the present article we want to put forward a few refinements of this model:

1) First we want to take into account that the number of different Feynman diagrams that can contribute to a given mass matrix element—a given effective Higgs-Yukawa coupling—becomes rather large when the diagram is complicated, because it involves many Higgs field-lines. This number of diagrams can be expressed in terms of factorials and can easily be so large that, even for order of magnitude calculations, we should include them.

2) In our previous work on this AGUT-fermion mass model, we introduced a Higgs field S for which we could take the vacuum expectation value (VEV) in the fundamental (Planck) units to be of order unity. Inclusion of the factorial corrections, just mentioned under point 1), actually makes the approximation of the S field VEV by unity give a poorer fit to the data. So, in order to regain as good a fit as before, we are forced to allow the VEV of the Higgs field S to be a variable parameter when the factorials are included; although its fitted value still turns out to be rather close to unity.

3) If there are many types of particles in the model, or if we have particles that like quarks are say triplets under colour symmetry², one would a priori imagine that the number of particles or the number of colours—i.e. 3—were tacitly treated as being of order unity in our previous fits of the quark and lepton masses. If this were really so, one would expect corrections of the order of, say, a factor 3 in our predictions. Actually our previous fits [1] always agreed with data to better than a factor 3, except for perhaps the most sensitive of our predictions—the CP-violation strength—which is especially uncertain, because it is a product of very many only order of magnitudewise known factors; in fact it was not included in the fit but simply predicted by our model. Even this

¹We do not discuss the neutrino mass problem here, which requires the introduction of another mass scale and corresponding Higgs particle.

²In the AGUT model there is even a different colour group for each proto-generation.

deviation of our CP-violation prediction from the experimental number is only just about a factor 3. In order to make sense of such a good agreement for our previous fits, we have to formulate a more precise form of the loose statement that the coupling constants are all of order unity at the Planck scale; so that we work to an accuracy where the number of colours (for the three SU(3) gauge groups in the model) and the number of species of particles, with masses of order the Planck scale, become large compared to the “order of magnitude accuracy” to which we hope to fit. We must hope—and indeed the estimates in section 6 suggest the hope is fulfilled—that we do not get large factors depending on the number of Planck scale species postulated in the model, because we do not know how many they should be. Rather we must show that their number is not important, provided they are at least there at all.

It is the purpose of the present article to include the two first improvements 1) and 2), and to discuss the third potential correction 3) which, however, turns out to actually only give somewhat milder corrections. For example we show that our prediction of order of magnitude degeneracy for quark and lepton masses in the same generation (except for the t and c quarks) remains valid, without any order $N_c = 3$ factors. So it becomes understandable that we can get order of magnitude fits with only about 40% deviations from the data.

Once we embark on including factors of suppression connected with our Higgs field S , which were previously set equal to unity, it matters how many times the nonzero VEV of this field S is needed to generate a given mass matrix element (or effective Yukawa coupling). The required number of factors of this VEV in turn depends on the gauge quantum numbers of the other AGUT Higgs fields, which we called W , T and ξ . So their quantum numbers must now be specified completely and not just modulo the quantum numbers of our Higgs field S , as was sufficient as long as the VEV of the S field could be considered to be of order unity. This specification of the precise gauge quantum numbers for the Higgs fields W , T , ξ and the field Φ_{WS} , containing the phenomenological Weinberg Salam Higgs field, gives rise to an infinite number of models. However, one should first of all remark that it is extremely reasonable to assume that, in a specific version of the model, the charges can only take on discrete—essentially integer in appropriate units—values, and secondly we do not expect them to be very large. So, at the end, the number of models to be considered is rather limited. We shall label the various considered models by a set of integers $(\alpha, \beta, \gamma, \delta)$ defined below in section 4.

In the following section 2 we give a short review of the AGUT model. Then, in the next section 3, we explain and implement the factorial correction coming from point 1): the large numbers of permutations of the Higgs fields in the Feynman diagrams. In section 4 we then define the set of models obtained by the various specific quantum number choices for our Higgs fields, i.e. by choosing $(\alpha, \beta, \gamma, \delta)$. In section 5 we present the results of fitting quark and lepton masses and mixing angles with the various parameter choices and with the factorial corrections included. Also discussed in this section are two variants

of how one interprets the order of unity factors in the calculation of the predicted masses etc.: one can either just put random phase factors on the products of the suppression factors and the factorial corrections, or one can further provide them with a random factor of order unity. The latter variation is not so important when one term dominates but when, as for the case of the d-quark mass in our model, more than one term contributes, it may give a correction. The best fitting Higgs field quantum number combinations are selected. Then follows, in section 6, our proposal for making the concept of the couplings being of order unity more precise. The outcome of this discussion is really that the results of section 5 are not significantly modified by the more careful definition of what the couplings being of order unity means. In section 7 we present the argument that, since many Feynman diagrams contribute, each mass matrix element in our model should have a Gaussian distribution in the complex plane. This, in turn, has the significance that we can even calculate an expectation for what the random element in the masses should be and, thus, predict the degree of deviation between our fit and the experimental numbers. Indeed we shall see that our prediction for the uncertainty agrees well with the experimental deviation of the fit. Finally in section 8 we present our conclusions.

2 Our AGUT-model, a short review

Anti-grand unification is a well suited name for our AGUT model in a number of senses, as we will now explain. It is similar to the usual grand unified theories (GUTs), which are based on $SU(5)$, $SO(10)$ or some other simple group, in the sense that we postulate the extension of the Standard Model Group [3], $SMG = S(U(2) \times U(3)) \approx U(1) \times SU(2) \times SU(3)$, to a larger gauge group at a very high energy of order the Planck scale. However the AGUT gauge group can be considered “anti”, as it is not at all simple in the mathematical sense of the word. On the contrary, its Lie algebra consists altogether of ten cross product factors: four $U(1)$ factors, three $SU(2)$ factors and three $SU(3)$ factors. Below the AGUT scale there is no supersymmetry nor other new physics beyond the Standard Model.

Further the usual GUTs unify by combining several Standard Model Group quark and lepton irreducible representations into the same irreducible representation of the grand unified group. Our AGUT is “anti” in the sense that it is characterised by not uniting any of the Standard Model Group irreducible representations into a larger irreducible representation. In this way our model avoids the exact mass degeneracies predicted from the minimal $SU(5)$ GUT, according to which the running masses of the quarks with charge $-1/3$ are degenerate with their charged lepton partners at the unification scale. Honestly speaking, except possibly for the case of the b quark being degenerate with the τ lepton, these predictions are wrong. In fact the unwanted predictions, $m_s = m_\mu$ and $m_d = m_e$, can only be tolerated by complicating the $SU(5)$ model [4]. How-

ever it turns out that the AGUT model replaces these exact $SU(5)$ predictions by only order of magnitude degeneracy predictions, which are compatible with experiment.

While $SU(5)$ is the smallest simple group unifying quarks and leptons, the AGUT group $SMG^3 \times U(1)_f$ is the biggest group, which keeps the Standard Model irreducible representations separate, in the following sense: there is no larger group, containing $SMG^3 \times U(1)$ as a subgroup, which is faithfully represented on the Standard Model quark and lepton fields alone (no right-handed neutrinos or other new fermions), without gauge or mixed anomalies. Indeed the $SMG^3 \times U(1)$ group is uniquely specified as the group G extending the Standard Model group and satisfying the following four postulates [5].

1. G should transform the three generations of Standard Model Weyl particles into each other unitarily, so that $G \subseteq U(45)$.
2. G should be anomaly free, even without using the Green-Schwarz [6] anomaly cancellation mechanism.
3. The fifteen irreducible representations of the Standard Model Weyl fields remain irreducible under G .
4. G is the maximal group satisfying the other three postulates.

Also the Standard Model gauge coupling constants do not unify in the AGUT model, but their values have been successfully predicted, at the Planck scale, using the so-called multiple point principle [7]. According to this principle, the coupling constants are fixed at such values as to ensure the existence of many vacuum states with the same energy density. We note that the top quark mass and the Weinberg Salam Higgs particle mass can also be predicted using this principle [8].

Really the existence of this AGUT group means that, near the Planck scale, each of the three quark-lepton proto-generations has its own SMG gauge group and associated 12 gauge particles—i.e. the gauge bosons also come in generations. In addition there is an extra abelian $U(1)_f$ gauge boson; the corresponding gauge charge Q_f is not carried by the left-handed fermions or any of the first proto-generation particles. So the non-zero $U(1)_f$ quantum numbers for the proto- quarks and leptons can be chosen as follows:

$$Q_f(\tau_R) = Q_f(b_R) = Q_f(c_R) = 1 \quad (1)$$

$$Q_f(\mu_R) = Q_f(s_R) = Q_f(t_R) = -1. \quad (2)$$

We stress that the physical quarks and leptons are superpositions of the proto-fermions. In particular it turns out that the right-handed charm and top quarks are essentially permuted: the physical right-handed top quark is the Weyl component with second proto-generation quantum numbers, namely the right-handed proto-charm quark, to first approximation.

The representations of the Standard Model Group $SMG = S(U(2) \times U(3))$ satisfy the charge quantization rule:

$$y/2 + d/2 + t/3 = 0 \pmod{1} \quad (3)$$

Here y is the conventional weak hypercharge and d is the duality, which is defined to be 0 when the weak isospin is integer and $d = 1$ when it is half integer. Similarly the triality t is defined to be 0 for an $SU(3)$ colour singlet and $t = \pm 1$ for a triplet or antitriplet respectively. In the AGUT model this charge quantisation rule is satisfied separately for the SMG quantum numbers associated with each quark-lepton generation.

Now we turn to the breaking of the AGUT gauge symmetry group $SMG^3 \times U(1)_f$ down to the Standard Model group SMG , which is supposed to occur close to the Planck scale. Unlike for the quarks and leptons, the gauge quantum numbers of the Higgs fields responsible for this symmetry breaking are not determined by the theoretical structure of the model. We assume that these Higgs fields have “small” quantum numbers, belonging to singlet or fundamental representations of the non-Abelian groups. The charge quantisation rule, Eq. (3), then determines the non-Abelian quantum numbers from the Abelian ones. So it is sufficient to specify just the Abelian quantum numbers in the form of a $U(1)$ charge vector:

$$\mathbf{Q} \equiv \left(\frac{y_1}{2}, \frac{y_2}{2}, \frac{y_3}{2}, Q_f \right) \quad (4)$$

where y_i is the conventional weak hypercharge for the i 'th proto-generation. For example the Higgs field S , introduced in [1] with a VEV of order unity in fundamental units, has the Abelian quantum numbers $\mathbf{Q}_S = (\frac{1}{6}, -\frac{1}{6}, 0, -1)$. It follows from the charge quantisation rule that the full set of AGUT quantum numbers for the S field are: $(\mathbf{3}, \mathbf{2}, \frac{1}{6}; \bar{\mathbf{3}}, \mathbf{2}, -\frac{1}{6}; \mathbf{1}, \mathbf{1}, 0; -1)$, where the sets of three quantum numbers specifying the representations under the three proto-generation SMG -groups are separated by semi-colons.

As long as we consider the VEV of the Higgs field S to be of order unity, we are in reality working with the group to which S breaks the AGUT group down. Then it is only necessary to specify the quantum numbers under this subgroup (i.e. modulo the quantum numbers of S). Thus the quantum numbers $\mathbf{Q}_T = (0, -\frac{1}{6}, \frac{1}{6}, -\frac{2}{3})$ for the Higgs field T used in [1] could just as well be taken to be $\mathbf{Q}_T = (-\frac{1}{6}, 0, \frac{1}{6}, \frac{1}{3})$ modulo the S quantum numbers. This transformed set is of course arbitrary as far as adding S quantum numbers is concerned, but we have chosen it as in some sense the smallest Abelian quantum number choice.

Another crucial assumption in our model is the existence, at the fundamental (Planck) scale, of a large spectrum of vector-like Dirac particles with the quantum numbers needed in the Feynman diagrams generating the quark-lepton mass matrices [2]. Furthermore it is assumed that all the couplings, allowed by the AGUT gauge symmetries, of these Planck scale particles to each other and to the lighter Standard Model particles are of order unity.

In practice we construct the mass matrix elements between the known Standard Model left- and right-handed Weyl fields one by one, by finding out first which quantum numbers need to be broken to make a given matrix element different from zero. Knowing the needed breaking quantum numbers, we then ask which combination of Higgs fields in our model can together cause that breaking. Measuring everything in “fundamental” units we can say that, since everything is of order unity except the small VEVs, we just take the product of the VEVs for the needed combination of Higgs fields. Finally the mass matrix element in question is simply order of magnitudewise equal to this product. However we should mention that there is an ambiguity concerning which combination of Higgs fields to use, since the Abelian quantum numbers of our proposed Higgs fields W , T and ξ obey a linear relation: $3\mathbf{Q}_W - 9\mathbf{Q}_T - 2\mathbf{Q}_\xi = 0 \pmod{S}$ (quantum numbers). The resolution of this ambiguity is to choose the combination of Higgs field VEVs giving the largest value for the product. There is no problem in practice due to the large powers of T involved.

3 Combinatorial corrections to mass matrices

We will now discuss a modification of the fitting procedure we have used previously [1]. The elements of the mass matrices are determined [2] up to order of one factors to be a product of several Higgs vacuum expectation values (measured in units of some fundamental scale M_F which we will take to be the Planck scale.) This is because the entries in the mass matrices come from Feynman diagrams such as that shown in Fig. 1. We have used such mass matrices to fit the fermion masses and mixing angles. In fact we have managed to produce fits with a smaller χ^2 than would be expected, given the lack of knowledge of order of 1 factors. So it is sensible to try to make a more refined fit, which takes into account systematic factors, even though we will still have order of 1 uncertainties which we can only average over.

One effect we have ignored up till now are the combinatorial factors due to the internal ordering in the Feynman diagrams. That is, we have only taken a single Feynman diagram to calculate the order of magnitude of each mass matrix entry. What we should really do is sum up all Feynman diagrams with given initial and final states. If we restrict ourselves to tree level diagrams this means that we should consider the sum over all distinct orderings of the interactions with the Higgs VEVs. If we consider a diagram with order of magnitude $\langle W \rangle^a \langle T \rangle^b \langle \xi \rangle^c \langle S \rangle^d \langle \Phi_{WS} \rangle$, then the corresponding matrix element should now be calculated by summing over all $\frac{(a+b+c+d+1)!}{a!b!c!d!}$ diagrams. Since in our approach only the order of magnitude of each diagram is determined, we should add all these diagrams together with random phases. Therefore the order of magnitude of this matrix element is given by $\left[\frac{(a+b+c+d+1)!}{a!b!c!d!} \right]^{\frac{1}{2}} \langle W \rangle^a \langle T \rangle^b \langle \xi \rangle^c \langle S \rangle^d \langle \Phi_{WS} \rangle$.

It would seem at first sight that these combinatorial factors would greatly

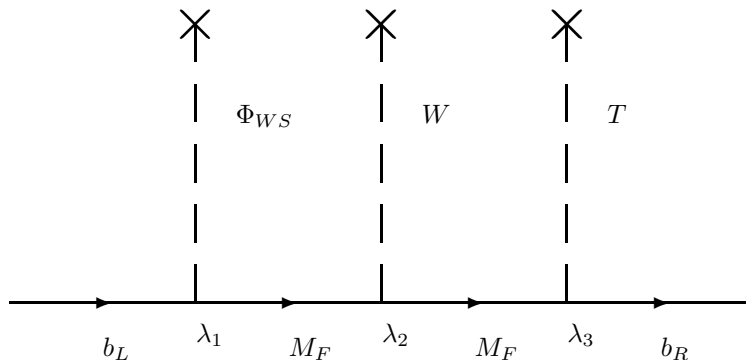


Figure 1: Possible Feynman diagram for bottom quark mass. The crosses indicate the couplings of the Higgs fields to the vacuum. Note that this is only 1 of 6 possible tree level diagrams since the order of the interactions is arbitrary.

modify our mass matrices and so vastly alter the fits we have already analysed without such factors. However, this is not quite true since the factors can to a large extent be absorbed into the Higgs VEVs. This means that we can get similar fits, with or without such factors, but with rescaled Higgs VEVs. But of course there will still be some differences due to these factors, since they cannot be exactly absorbed by redefining the Higgs VEVs. One important point is that now even if we assume that the Higgs field S has VEV $\langle S \rangle = 1$, this field still affects the fit through its contribution to the combinatorial factors. So, whereas previously the quantum numbers of such a Higgs field could be freely absorbed by other Higgs fields without changing anything, we will now get slightly different fits depending on the precise definition of the quantum numbers of the Higgs fields.

4 Mass matrices and Higgs quantum numbers

The quantum numbers of the Higgs fields S , W , T , ξ and Φ_{WS} were originally constructed [1] by requiring them to give phenomenologically acceptable relations between the lepton and quark masses and their mixing angles. As we have already emphasized, the quantum numbers of the W , T , ξ and Φ_{WS} fields were only determined modulo those of the S field. We here arbitrarily select a standard set of such Higgs field quantum numbers, with “small” values, and parameterise the equivalent sets of quantum numbers by integer parameters $(\alpha, \beta, \gamma, \delta)$, defining how many S quantum numbers have been added. This leads to the Higgs fields having the following charges:

$$Q_S = \left(\frac{1}{6}, -\frac{1}{6}, 0, -1\right) \quad (5)$$

$$Q_W = \left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{2}, -\frac{1}{3}\right) + \alpha Q_S \quad (6)$$

$$Q_T = \left(-\frac{1}{6}, 0, \frac{1}{6}, \frac{1}{3}\right) + \beta Q_S \quad (7)$$

$$Q_\xi = \left(\frac{1}{6}, -\frac{1}{6}, 0, 0\right) + \gamma Q_S \quad (8)$$

$$Q_{WS} = \left(\frac{1}{6}, \frac{1}{2}, -\frac{1}{6}, 0\right) + \delta Q_S \quad (9)$$

The orders of magnitude of the up quark, down quark and charged lepton mass matrices are given in terms of the Higgs fields as follows:

$$M_U = H_U \Phi_{WS}^\dagger \quad (10)$$

$$M_D = H_D \Phi_{WS} \quad (11)$$

$$M_E = H_E \Phi_{WS} \quad (12)$$

where the order of magnitude of the Yukawa coupling matrix H_U is given by:

$$\begin{pmatrix} S^{1+\alpha-2\beta+2\gamma+\delta} W^\dagger T^2 (\xi^\dagger)^2 & S^{2+\alpha-2\beta-\gamma+\delta} W^\dagger T^2 \xi & S^{2+\alpha-\beta-\gamma+\delta} (W^\dagger)^2 T \xi \\ S^{1+\alpha-2\beta+3\gamma+\delta} W^\dagger T^2 (\xi^\dagger)^3 & S^{2+\alpha-2\beta+\delta} W^\dagger T^2 & S^{2\alpha-\beta+\delta} (W^\dagger)^2 T \\ S^{3\gamma+\delta} (\xi^\dagger)^3 & S^{1+\delta} & S^{-1+\alpha+\beta+\delta} W^\dagger T^\dagger \end{pmatrix}$$

the order of magnitude of the Yukawa coupling matrix H_D is given by:

$$\begin{pmatrix} S^{-1-\alpha+2\beta-2\gamma-\delta} W (T^\dagger)^2 \xi^2 & S^{-2-\alpha+2\beta-\gamma-\delta} W (T^\dagger)^2 \xi & S^{2-3\beta-\gamma-\delta} T^3 \xi \\ S^{-1-\alpha+2\beta-\gamma-\delta} W (T^\dagger)^2 \xi & S^{-2-\alpha+2\beta-\delta} W (T^\dagger)^2 & S^{2-3\beta-\delta} T^3 \\ S^{-2-2\alpha+4\beta-\gamma-\delta} W^2 (T^\dagger)^4 \xi & S^{-3-2\alpha+4\beta-\delta} W^2 (T^\dagger)^4 & S^{1-\alpha-\beta-\delta} W T \end{pmatrix}$$

and the order of magnitude of the Yukawa coupling matrix H_E is given by:

$$\begin{pmatrix} S^{-1-\alpha+2\beta-2\gamma-\delta} W (T^\dagger)^2 \xi^2 & S^{-2-\alpha+2\beta+3\gamma-\delta} W (T^\dagger)^2 (\xi^\dagger)^3 & S^{2-\alpha-4\beta+\gamma-\delta} W T^4 \xi^\dagger \\ S^{-1-\alpha+2\beta-5\gamma-\delta} W (T^\dagger)^2 \xi^5 & S^{-2-\alpha+2\beta-\delta} W (T^\dagger)^2 & S^{2-\alpha-4\beta-2\gamma-\delta} W T^4 \xi^2 \\ S^{-2-\alpha+5\beta-3\gamma-\delta} W (T^\dagger)^5 \xi^3 & S^{1+2\alpha-4\beta-\delta} (W^\dagger)^2 T^4 & S^{1-\alpha-\beta-\delta} W T \end{pmatrix}$$

The entries in the mass matrices represent the Higgs fields involved in the Feynman diagram describing the tree-level interaction relevant for each element. It is to be understood that the actual values of the mass matrix elements are given by the products of the expectation values of the Higgs fields involved. So, for example, $(W^\dagger)^2$ would mean $\langle W \rangle^2$ and S^n means $\langle S \rangle^{|n|}$. Also, for simplicity, we haven't included the combinatorial factors in the mass matrices. These factors must be included in the fit and are easily calculated from the powers of each Higgs VEV, as in the previous section.

So the charges of the Higgs fields used in our previous paper [1] were given by the choice $\alpha = 1$, $\beta = 1$, $\gamma = 0$ and $\delta = -1$. However, without the combinatorial factors and with $\langle S \rangle = 1$, the choice of α , β , γ and δ does not affect the fit. But now we will include these factors and vary the integers α , β , γ and δ to see what the effect on the fit is. Clearly large values of these variables will introduce large powers of $\langle S \rangle$ and large combinatorial factors which will change the character of the mass matrices. Since we have derived the model under the assumption that $\langle S \rangle \approx 1$ and that the combinatorial factors are not very important, we should restrict ourselves to small values of the integer parameters α , β , γ and δ . Also this will satisfy our general requirement that the model should have “small” quantum numbers. We allow $\langle S \rangle$ to vary, in order to somewhat compensate for the combinatorial factors. We decided to allow values -1, 0 and 1 for each parameter α , β , γ and δ . This gives us a total of 81 choices. We want to find the best fit among these 81, minimising a pseudo-chisquared function defined in Eq. (13) below. To do this we first made an approximate fit for all 81 choices, giving an average pseudo-chisquared $\chi_{ave}^2 = 2.7$, and then chose those fits which had $\chi^2 < 2.0$. This left us with 14 fits, which we then analysed with higher accuracy.

5 Results

As in previous papers, we fit to the experimental values given in Table 1 and use a pseudo-chisquared function to measure how good a fit is. Since we are making an order of magnitude fit, we use the definition:

$$\chi^2 = \sum \left[\ln \left(\frac{m}{m_{\text{exp}}} \right) \right]^2 \quad (13)$$

where m are the fitted masses and mixing angles and m_{exp} are the corresponding experimental values in Table 1. The χ^2 was minimised by varying the Higgs VEVs, where in this paper we also vary $\langle S \rangle$ rather than fixing $\langle S \rangle = 1$. The Yukawa matrices are calculated at the fundamental scale, which we take to be the Planck scale. We use the first order renormalisation group equations (RGEs) for the Standard Model to calculate the matrices at lower scales. Running masses are calculated in terms of the Yukawa couplings at 1 GeV.

In this section we will comment on the fits in Tables 2 and 3, in order to highlight the general features of different choices for the charges of the Higgs fields. We will start by making a comparison between using random complex $O(1)$ factors in the fitting procedure or just random phase factors.

As we have repeatedly stressed, we are only assuming some knowledge of the order of magnitude of the mass matrix elements. This means that, when calculating the eigenvalues of these matrices, we should at least consider each element to have an arbitrary phase. We do this by averaging over the calculated eigenvalues, using many random choices of phases for all the elements.

Table 1: Experimental values of masses and mixing angles used in the fits. All masses are running masses at 1 GeV except the top quark mass which is the pole mass.

m_u	4 MeV	m_c	1.4 GeV
m_d	9 MeV	m_s	200 MeV
m_e	0.5 MeV	m_μ	105 MeV
M_t	180 GeV	V_{us}	0.22
m_b	6.3 GeV	V_{cb}	0.041
m_τ	1.78 GeV	V_{ub}	0.0035

Table 2: Results for fits without any O(1) factors used.

α	β	γ	δ	$\langle W \rangle$	$\langle T \rangle$	$\langle S \rangle$	$\langle \xi \rangle$	χ^2
-1	-1	-1	1	0.0672	0.0667	0.487	0.0331	1.81
-1	0	-1	1	0.0857	0.0522	0.33	0.0365	1.34
-1	0	1	-1	0.0705	0.0549	0.720	0.0422	1.88
-1	1	-1	1	0.0735	0.0525	0.686	0.0331	1.90
-1	1	1	1	0.0857	0.0498	0.720	0.0402	1.68
0	0	-1	0	0.0610	0.0851	0.286	0.0315	1.47
0	0	-1	1	0.0671	0.0576	0.464	0.0402	1.42
0	0	0	0	0.0705	0.0810	0.259	0.0365	2.04
0	1	1	0	0.0945	0.0522	0.876	0.0365	1.78
1	-1	-1	1	0.0740	0.0576	0.622	0.0315	1.91
1	-1	1	1	0.0777	0.0548	0.622	0.0402	2.03
1	0	-1	0	0.0777	0.0575	0.537	0.0443	1.87
1	0	-1	1	0.0740	0.0605	0.512	0.0331	1.94
1	1	-1	0	0.0992	0.0548	0.271	0.0489	1.68

Table 3: Results for fits including O(1) factors.

α	β	γ	δ	$\langle W \rangle$	$\langle T \rangle$	$\langle S \rangle$	$\langle \xi \rangle$	χ^2
-1	-1	-1	1	0.0741	0.0635	0.487	0.0331	1.57
-1	0	-1	1	0.0945	0.0522	0.347	0.0331	1.41
-1	0	1	-1	0.0857	0.0522	0.686	0.0365	1.59
-1	1	-1	1	0.0894	0.0525	0.756	0.0247	1.26
-1	1	1	1	0.0945	0.0474	0.653	0.0365	1.46
0	0	-1	0	0.0741	0.0810	0.286	0.0300	1.27
0	0	-1	1	0.0857	0.0548	0.442	0.0347	1.40
0	0	0	0	0.0816	0.0735	0.299	0.0331	1.37
0	1	1	0	0.0945	0.0522	0.721	0.0331	1.62
1	-1	-1	1	0.0857	0.0522	0.622	0.0300	1.44
1	-1	1	1	0.0900	0.0497	0.622	0.0383	1.70
1	0	-1	0	0.0900	0.0522	0.537	0.0422	1.79
1	0	-1	1	0.0816	0.0549	0.538	0.0331	1.64
1	1	-1	0	0.1042	0.0522	0.346	0.0444	1.85

Table 4: Typical fit without averaging over O(1) factors with $\alpha = -1$, $\beta = 1$, $\gamma = 1$ and $\delta = 1$. All masses are running masses at 1 GeV except the top quark mass which is the pole mass.

	Fitted	Experimental
m_u	4.2 MeV	4 MeV
m_d	4.7 MeV	9 MeV
m_e	0.98 MeV	0.5 MeV
m_c	1.22 GeV	1.4 GeV
m_s	340 MeV	200 MeV
m_μ	83 MeV	105 MeV
M_t	220 GeV	180 GeV
m_b	7.2 GeV	6.3 GeV
m_τ	1.17 GeV	1.78 GeV
V_{us}	0.15	0.22
V_{cb}	0.031	0.041
V_{ub}	0.0040	0.0035
J_{CP}	9.4×10^{-6}	$2 - 3.5 \times 10^{-5}$
χ^2	1.68	

Table 5: Typical fit including averaging over $O(1)$ factors with $\alpha = -1$, $\beta = 1$, $\gamma = 1$ and $\delta = 1$. All masses are running masses at 1 GeV except the top quark mass which is the pole mass.

	Fitted	Experimental
m_u	3.1 MeV	4 MeV
m_d	6.6 MeV	9 MeV
m_e	0.76 MeV	0.5 MeV
m_c	1.29 GeV	1.4 GeV
m_s	390 MeV	200 MeV
m_μ	85 MeV	105 MeV
M_t	179 GeV	180 GeV
m_b	7.8 GeV	6.3 GeV
m_τ	1.29 GeV	1.78 GeV
V_{us}	0.21	0.22
V_{cb}	0.023	0.041
V_{ub}	0.0050	0.0035
J_{CP}	1.04×10^{-5}	$2 - 3.5 \times 10^{-5}$
χ^2	1.46	

This avoids any accidental cancellations between quantities of the same order of magnitude. However, we can also consider introducing random factors which are of order 1, since this will not change the order of magnitude of each element. Now, of course, we must decide more precisely what we mean by an $O(1)$ factor and how to average over the random variations. There is no unique way to decide how large a number should be before we no longer consider it to be of order 1. However, a reasonable choice is to say that a real number is of order 1 if its natural logarithm lies between -1 and 1. So we will pick random $O(1)$ factors by taking the exponential of a number picked from a Gaussian distribution of width 1. We will now compare the results from fitting after averaging over these $O(1)$ factors to the results without using any $O(1)$ factors (but still averaging over random phases.) Tables 2 and 3 give the χ^2 values for the best 14 fits with and without $O(1)$ factors. However, in order to illustrate in detail the differences, we have shown the fitted masses and mixing angles for a typical fit in Tables 4 and 5.

The most obvious differences between using $O(1)$ factors or not is in the ratio between up and down quark masses. The difference between these masses is always increased when $O(1)$ factors are used. The reason for this is that the down mass is produced by two different combinations of matrix elements. Since the down mass will typically be given by the root mean square average of these two combinations, random factors will generally increase this mass more

than other masses which are essentially only determined by one combination of matrix elements. In most of the fits this helps but, in a few, the ratio is already large enough without the $O(1)$ factors and this makes the fit worse. The $O(1)$ factors also seem to allow a better fit for the electron mass, which is too high without the $O(1)$ factors. This is because the down mass can still be fitted with a lower electron mass (and so lower up mass), if the ratio of down:up masses is increased. It can be argued that we could adjust the spread of $O(1)$ factors to tune this ratio. Then this would introduce an extra parameter in our fit, since there is no good way to fix it. However, we would argue that we have chosen a ‘natural’ spread of these factors and that this spread can be considered, loosely speaking, as a definition of what we mean by $O(1)$.

The other effects of using $O(1)$ factors are less important. In the third generation the $O(1)$ factors reduce the top mass and increase the bottom mass, while leaving the tau mass almost unchanged. However, these are small effects although this does usually worsen the predictions. In the second generation the $O(1)$ factors increase the charm mass, which is still too low, but also increase the strange mass, which is always too high in our model. The muon mass is not changed much. These are again small effects and the improvement of the charm mass approximately cancels the worsening of the strange mass in the fit. However, this obviously increases the dominance of the contribution of the strange quark mass to the χ^2 for the fits. The $O(1)$ factors increase V_{us} which is still predicted too small. They decrease V_{cb} and this is usually bad but sometimes it is too high without $O(1)$ factors. V_{ub} is increased, which sometimes helps, but often it is predicted too small without $O(1)$ factors and too large with $O(1)$ factors so that, by chance, the $O(1)$ factors make almost no difference to the V_{ub} contribution to the value of χ^2 . Overall the $O(1)$ factors lead to a better prediction for the mixing angles and a small increase in CP violation. In general the $O(1)$ factors improve the fit, mainly due to the better fitting of the first generation masses and also the mixing angles V_{us} and V_{ub} . Of the 14 fits, only 2 are worse with the $O(1)$ factors. In both cases this is because the down:up mass ratio is already large enough without $O(1)$ factors. For $\alpha = -1$, $\beta = 0$, $\gamma = -1$ and $\delta = 1$ the fit is only slightly worse with $O(1)$ factors and is still a good fit, since the worse up mass is compensated for by a better electron mass and V_{us} . For $\alpha = 1$, $\beta = 1$, $\gamma = -1$ and $\delta = 0$ many of the general comments do not apply. Here the down:up mass ratio is surprisingly smaller with $O(1)$ factors and the mixing angles V_{us} and V_{ub} are almost unchanged. This leads to a worse fit with $O(1)$ factors, though we don’t know why the down:up mass ratio changes in this way for this case.

It is much harder to give any general comparison between the different choices of α , β , γ and δ . The values shown in Tables 2 and 3 have been chosen since we expected reasonably good fits from them. They are, in fact, the best fits out of all 81 combinations of α , β , γ and δ with values -1, 0 or 1. On general grounds we would expect worse fits from larger values of α , β , γ and δ . This is because large powers of the Higgs field $\langle S \rangle$ would be required for many matrix

elements. This would then suppress the elements unless $\langle S \rangle = 1$. We would also expect large contributions from the factorial factors, but it would be unlikely that these could be balanced by the suppression due to $\langle S \rangle$ for all matrix elements. Therefore we would end up with some elements larger than expected and others smaller than expected. Since these effects have not been taken into account in our derivation of the model, they would almost certainly spoil our assumed relations. We will now just consider the cases with small values of α , β , γ and δ that have been displayed in the Tables 2 and 3. We have displayed the fitted χ^2 for all the fits. Tables 4 and 5 show the fitted masses and mixing angles for typical cases, with and without averaging over $O(1)$ factors. Also shown is the predicted Jarlskog CP violation invariant J_{CP} [9].

We can see that generally the largest contributions to χ^2 come from the strange mass, up mass, electron mass, top mass and V_{us} . The strange mass is always fitted too high and is fitted the worst. It doesn't vary much among the fits, indicating that the fit minimises the strange mass since it is the largest single contribution to χ^2 . Therefore the results clearly show that the strange mass cannot be accurately fitted within our model, no matter what values of α , β , γ and δ are chosen. So we are left with the conclusion that the strange mass must be accidentally small, if our model is to be believed. This is not necessarily a problem, since we only claim to fit order of magnitudes and a factor of 2 is not so large.

We can see some trends in the way other masses and mixing angles are fitted with different values of α , β , γ and δ . There is generally a compromise between the up and electron masses (up too low and electron too high). The down mass is then largely determined by the ratio to the up mass, which varies between fits. This ratio depends mainly on γ —certainly it seems that $\gamma = -1$ gives a better ratio than $\gamma = 1$. This is presumably the main reason why most of the best fits (these 14 were the best out of 81) have $\gamma = -1$ and in fact the two very best fits both have $\gamma = -1$. Unfortunately there doesn't seem to be any simple dependence on α , β , γ or δ for the predicted top mass or V_{us} . Since many of the fits are almost equally good, there is probably some 'unpredictable' dependence on the factorials which makes it harder to determine such a dependence. Since so many fits are fairly good, it seems that we have little chance of really choosing one scheme as the best. This is perhaps to be expected, since the choices of α , β , γ and δ are in some sense just a fine-tuning of the general model. The model was 'derived' to include important relations between observed masses and mixing angles. Therefore, provided these variations don't spoil such assumed relations, we would not expect great differences in the overall fit. This is perhaps most obvious in that the strange mass is fitted 'equally badly' in the different variations of the model.

6 Making order unity concept more precise

We shall now take into account how to define “order of unity” more precisely, when the number of particle species, or the dimension of some representation to which they belong, is so large that we do not want to consider it of “order unity”. We shall, however, formally proceed in the estimate below as if the number of lines etc. in a Feynman diagram can be considered of “order unity”; in particular we shall not consider the number of external lines to be more than of order unity.

In the crudest approximation the following assumption should make sense:

Assumption A: At the Planck scale all masses and couplings are of order unity and particles with all required quantum numbers—namely all quantum numbers possible—do exist.

However we should like to work to a level of accuracy where, for instance, the number of colours N_c —which is say 3—can no longer be counted as of order unity, but rather should be considered much bigger than unity. It is then not a priori obvious whether the above crude “order unity” approximation can even be made consistent to such an accuracy nor, more precisely, how it should be done.

We here propose the following criterion for a successful more precise specification of the order unity of couplings concept:

The criterion: The main guideline we shall use is the requirement that composing two order unity amplitudes, say one N -point function and one P -point function to one $(N+P-2)$ -point function by combining via a propagator, should result in an order unity composite amplitude. This propagator is here assumed to only include particles that are not mass protected, so that they have Planck order of magnitude masses.

The requirement that we only include propagators for particles which are not mass protected reflects the fact that, in assumption A, we “assumed the order unity to be valid for the particles at the Planck scale”. It means that the quarks or leptons themselves are not supposed to be included as propagator particles in this criterion. What we calculate are amplitudes due only to exchanges of heavy (Planck scale) particles, and all the quark or lepton propagators must be inserted explicitly later (after the use of the criterion and the construction of the Planck scale particle exchange amplitudes). An example of such a later insertion can be found in subsection 6.4.

Let us argue now for why we should take this criterion to be obeyed as a test for a successful implementation of assumption A:

Consider an $(N + P - 2)$ -point function. In tree diagram approximation we can ask for the contributions to this $(N + P - 2)$ -point function coming

from diagrams with a propagator in a certain channel specified by, say, $N - 1$ of the external lines. For each combination of $N - 1$ lines out of the $N + P - 2$ external lines we have a contribution. The statement “particles with all required quantum numbers do exist” may be taken to suggest that none of these contributions are allowed to be much smaller than the other ones (i.e. we take them all to be of the same order of magnitude). Now such a contribution associated with a certain channel, in the sense of a selection of $N - 1$ of the external lines, can be written as a P -point function connected by a propagator to an N -point function. So such a single contribution is given by N -point and P -point functions with a propagator connecting them. Since we take all the contributions equal in size, order of magnitudewise, and since there are in principle “only a few” (because we assumed that the N etc. of the N -point function is not large) they must all—order of magnitudewise—coincide with the full amplitude, the $(N + P - 2)$ -point amplitude. But that means that we argued for the criterion to be fulfilled, in order that the assumption A can be said to be true.

In the above argument we used the approximation—which is perhaps not so good in practice—that the number of external particles in the considered amplitudes were of order unity with sufficient accuracy. For the application to the quark and lepton mass matrices, this number is equal to the number of Higgs fields used plus 2 for the quarks or leptons. However, we could say that the “factorial corrections” treated at length in section 3—the main new feature in the fits presented in this article—were precisely to take into account factors arising from the number N for these amplitudes not being of order unity accurately enough. Actually it turned out that the effects we studied were the $\sqrt{N!}$ factors, coming from including diagrams with the internal channels used being changed corresponding to the permutation of the external lines (or rather some of them).

We shall now explain how reasonable “order unity” N -point amplitudes are proposed, and subsequently confronted with our criterion. First we shall, for simplicity, discuss the case of there being no conservation laws, so that there are no forbidden couplings.

6.1 Case of no conservation laws

In order to define what the “order unity” should mean for the N -point amplitude, it is convenient to consider Euclidean momenta, since then the poles in the amplitude are avoided. The natural choice of definition is then that the N -point function for an arbitrary set of particles, with Euclidean momenta of order unity in fundamental units, shall be of order unity in the simple mathematical sense:

$$A(p_1, p_2, \dots, p_N) \approx 1 \quad (\text{in Planck units}). \quad (14)$$

This is indeed what we shall propose, but there are a few things to have in mind here:

In usual notations, it is customary to use propagators for which the residues of “the” pole are normalized. But the propagator is the inverse—as a matrix—of the two-point function, and if we thus use our specification, Eq. (14), to determine even the two-point functions, we loose the freedom to normalize the residues.

Also we shall have in mind that there are many species of particles of the Planck scale mass type (we here do not include the mass protected quarks and leptons) and that the two-point functions make up a matrix, the inverse of which is the propagator, which also is a matrix. Our postulate, Eq. (14), means that the individual matrix elements in the two-point function are just of order unity. Thus the species, in terms of which we express the order unity amplitudes are NOT mass eigenstates³. We shall make these considerations of the order unity calculations more precise statistically, by assuming that we have a statistical ensemble of order unity coefficients in the Lagrangian written in terms of fields for the various species, with a spread of order unity (in Planck units) so that they are practically always of order unity. Since the free terms—the kinetic term as well as the mass term—are then random, these species end up being rather random compared to the mass eigenstates of course.

The propagator is then easily estimated to have matrix elements of the order $1/(\# \text{ of species})$. This may be seen by arguing that the eigenvalues of the inverse propagator are of the order $\sqrt{\# \text{ of species}}$, while the eigenvalues of the propagator must of course then be of order $1/\sqrt{(\# \text{ of species})}$. So the propagator is a factor of $(\# \text{ of species})$ smaller than the inverse propagator.

Provided we use the propagator as the inverse of the two-point function matrix, it does not really matter if we multiplied all the N point functions by the N th power of any number (e.g. the square root of the number of species $\sqrt{\# \text{ of species}}$). This is because such a factor for each line would be cancelled by the propagators, or in the case of external lines by the wave function (re)normalization. So actually we could equivalently have postulated the “order unity” rule as:

$$A(p_1, p_2, \dots, p_N) \approx 1/(\# \text{ of species})^{(N/2)} \quad (\text{in Planck units}) \quad (15)$$

This really may also be described as saying that the sum of the numerical squares of A over all combinations of external particles should be of the order of unity.

But, before we accept one of these (essentially equivalent) ansätze, we should check that the criterion is fulfilled:

In order to check the criterion, we write down the double sum⁴ over the species at the end-points of the propagator connecting the N -point and P -point functions to form a composite $(N+P-2)$ -function. Now, since there are

³If wanted, the mass eigenstates would have to be found by dividing the mass term, as a matrix, by the kinetic term, also a matrix, and then diagonalizing the resulting matrix.

⁴It should be remembered here that our species were neither mass nor kinetic term coefficient eigenstates so that a DOUBLE sum is needed, the propagator being a matrix.

(# of species) terms in each of the two summations and we take each term to be a complex random number, these sums effectively each function as a factor of $\sqrt{\# \text{ of species}}$. Since the matrix elements of the propagator are of the order $(1/\# \text{ of species})$ in the notation of Eq. (14) and unity in the notation of Eq. (15), we end up in both cases with a composite amplitude of the correct “order of unity”.

6.2 Case with e.g. $SU(N_c)$ symmetry

In the case where there is a symmetry, like colour symmetry under the group $SU(N_c)$ say, we cannot simply use the rule for the case of no conservation laws given above, because a random N-point function would of course almost certainly violate the symmetry. Instead let us make the “order unity” assumption for N-point functions, by defining a probability distribution for amplitudes. Then we can say, in a precise sense, that an amplitude is of “order unity” when it is of a type that can be gotten with high probability from this distribution. The probability distribution is then specified as we describe below.

We first write down the cartesian product of the states of the N external particles counted as incoming (outgoing particles replaced by incoming antiparticles):

$$|sp1, c1\rangle |sp2, c2\rangle \cdots |spN, cN\rangle \in \mathcal{H}_N. \quad (16)$$

Here the symbols “ $sp1$ ” ... “ spN ” stand for the species (= “flavour”) of the external particles, while “ $c1$ ”...“ cN ” denote their colour states. Often one imagines working with (external) particle states $|sp1, c1\rangle, \dots |spN, cN\rangle$ that have, for instance, a definite weak isospin component along a certain quantization axis or definite values of some analogous colour Cartan algebra generators. However, for the purpose of formulating the statistical rules defining the “order of unity”, we shall here rather assume that the external states are random superpositions of various, e.g. colour and flavour, components. W.r.t. “flavours” (meant in the very general sense in which it is used here), it does not matter much to have this in mind, but for the conserved colour we risk to get many zeroes alternating with too big numbers for the amplitudes A_N , if we do not work with random states. So the following formulae are only meant to be valid for states that are random superpositions! An N-point function conserving colour is given by specifying a bra-vector $\langle R_N|$, belonging to the colour singlet subspace of the cartesian product Hilbert space \mathcal{H}_N (see Eq. (16)), as the overlap

$$A_N(p_1, sp1; p_2, sp2; \dots; p_N, spN) = \langle R_N(p_1, p_2, \dots, p_N) | |sp1, c1\rangle |sp2, c2\rangle \cdots |spN, cN\rangle. \quad (17)$$

We now make a random “order unity” N-point function by choosing this bra-vector $\langle R_N|$ randomly. We decide to take a “rotational invariant” direction distribution for it and a norm of the order of the square root of the dimension

of the space of singlets:

$$|| < R_N || \approx \sqrt{\dim(\text{space of singlets})} \quad (18)$$

This latter choice may a priori look rather arbitrary, but note that it is just the choice that corresponds to the overlap size of $< R_N |$ with any normalized singlet state $|sing\rangle$ in \mathcal{H}_N becoming of order unity:

$$< | < R_N | sing \rangle |^2 >_{R_N-\text{averaging}} \approx 1 \quad (19)$$

Really this statistical definition of the order unity amplitudes is a bit superfluous and we could more simply just state that the amplitude A_N becomes of the order of

$$A_N(p_1, sp1; p_2, sp2; \dots; p_N, spN) = \sum_{\text{singlet}_N} \text{"O(1) - factor"} < \text{singlet}_N | sp1, c1 \rangle | sp2, c2 \rangle \dots | spN, cN \rangle. \quad (20)$$

where we sum over a basis set of singlet states. We should here comment on the normalisation of the states w.r.t. colour and flavour (i.e. species). We are using the normalisation of Eq. (14) for the species degree of freedom. However we have chosen to use the normalisation of Eq. (15) for colour in the sense that, if colour conservation were broken without changing their order of magnitude, the amplitudes would be smeared out so that the colours would function as species with the normalisation of Eq. (15). In the case of no (colour) conservation laws discussed in subsection 6.1, the propagator with the notation of Eq. (15) would be just of order unity. So it is expected that the colour part of the propagator should on the average be of order unity. Indeed in the case of a colour conservation law, as considered in this subsection, we can easily estimate the propagator as the inverse of the two-point function A_2 . With external states belonging to a colour representation \underline{a} of dimension $n = \dim(\underline{a})$ and having colour components \vec{m} and \vec{m}' respectively, we obtain:

$$\left(< \vec{m}, spN | A_2 | \vec{m}', spP' \rangle \right)^{-1} \quad (21)$$

$$\approx \left(O(1)_{spN, spP'} < \underline{1}, \vec{0} | \underline{a}, \vec{m}; \underline{a}, -\vec{m}' \rangle \right)^{-1} \quad (22)$$

$$\approx \left(\frac{O(1)_{spN, spP'}}{\sqrt{\dim(\underline{a})}} \delta_{\vec{m}, \vec{m}'} \right)^{-1} \quad (23)$$

$$\approx O(1)_{spN, spP'} \frac{\sqrt{n}}{\# \text{ of species}} \delta_{\vec{m}, \vec{m}'} \quad (24)$$

$$\approx Prop_{\underline{a}; spN, spP'}(p_N) \delta_{\vec{m}, \vec{m}'} \quad (25)$$

where then⁵

$$Prop_{\underline{a}; spN, spP'}(p_N) \approx O(1)_{spN, spP'} \frac{\sqrt{n}}{\# \text{ of species}} \quad (26)$$

Now we want to show that our order unity amplitude definition, Eq. (20), has the nice property of passing the test of being self-consistent under composition of an N-point function and a P-point function by propagator contraction. We write down the composite (N+P-2)-point function formed from the propagator contracted N-point and P-point functions as follows:

$$\sum_{spN, spP', \underline{a}} A_N(p_1, sp1; p_2, sp2; \dots; p_N, spN) Prop_{\underline{a}; spN, spP'}(p_N) A_P(p'_1, sp1'; \dots; p'_P = p_N, spP') \quad (27)$$

Here $p_1, p_2, \dots, p_{N-1}, p'_1, p'_2, \dots, p'_{P-1}$ are the four-momenta of the external particles, and the first N-1 are attached to the N-point blob, while the last—and primed ones—are attached to the P-point blob. Furthermore sp stands for species or flavour with an analogous enumeration, and $Prop_{\underline{a}; spN, spP'}(p_N)$ is the propagator, which is a matrix, Eq. (26), in flavour or species space. The momentum of the propagator, determined from four-momentum conservation, is $p_N = p'_P$. The triple sum runs over all pairs of species or flavours for each colour representation \underline{a} in the sum.

The “self-consistency” we want to show is that this expression, Eq. (27), equals $A_{N+P-2}(p_1, sp1; p_2, sp2; \dots; p_{N-1}, sp(N-1); p'_1, sp1'; \dots; p'_{P-1}, sp(P-1)')$ order of magnitudewise, or rather that the statistical distributions are roughly equal for the two expressions.

According to the property, Eq. (19), the (N+P-2)-point function is of the form $< R_{N+P-2} | sp1, c1 > | sp2, c2 > \dots | sp(N-1), c(N-1) > | sp1', c1' > \dots | sp(P-1)', c(P-1)' >$, with $< R_{N+P-2} |$ having projection on average unity (order of magnitudewise) on all the singlet states $|sing >$. So we would be well on the way to proving the self-consistency condition, if we could show that the dimension of this singlet space were the same as that of the singlet space for the composed amplitude of Eq. (27). So as an introduction to prove the remarkable “self-consistency”

$$A_{N+P-2}(p_1, sp1; p_2, sp2; \dots; p_{N-1}, sp(N-1); p'_1, sp1'; \dots; p'_{P-1}, sp(P-1)') = \sum_{spN, spP', \underline{a}} A_N(p_1, \dots; p_N, spN) Prop_{\underline{a}; spN, spP'}(p_N) A_P(p'_1, \dots; p'_P = p_N, spP') \quad (28)$$

we remark that the number of colour contractions—i.e. of the singlets that can be used—is indeed the same for both sides of this equation.

⁵Of course the order unity numbers $O(1)_{spN, spP'}$ depend on spN and spP' , but they are just of order unity. So we could logically just have left the indices out and written $O(1)$ for $O(1)_{spN, spP'}$.

This can be seen in the following way: Imagine the singlets “of the left hand side” constructed by first coupling the colour representations of the $(N-1)$ and of the $(P-1)$ external lines of the $(N+P-2)$ -point function to respectively the \underline{a} and $\underline{\bar{a}}$ representations, finally collecting the singlets for the various \underline{a} -choices. Classified this way it is easily seen that these singlets are in correspondence with “those of the right hand side”, since the \underline{a} -classification corresponds to counting according to the representation \underline{a} of the propagators. Thus we have the equation for the number of singlets:

$$\#singlets_{N+P-2} = \sum_{\underline{a}} \#singlets_N(\underline{a}) * \#singlets_P(\underline{\bar{a}}) \quad (29)$$

where the representations \underline{a} in the expressions $\#singlets_N(\underline{a})$ and $\#singlets_P(\underline{\bar{a}})$ just denote the representations to which the species, spN and spP' respectively, should belong.

After having noticed the natural correspondence between the singlet components on the two sides of the to be proven equation, Eq. (28), we go over to considering a single contribution—one of the $\#singlets_{N+P-2}$ in Eq. (29)—to A_{N+P-2} . A contribution from a single “ $|singlet\rangle$ ” out of the $\#singlets_{N+P-2}$, called say s , to A_{N+P-2} (the s -given representation \underline{a} in the channel between the $P-1$ and the $N-1$ has $dim(\underline{a}) = n$) is

$$\begin{aligned} A_{N+P-2}^s = & \langle singlet(s) | sp1, p_1, c1 \rangle | sp2, p_2, c2 \rangle \cdots | sp(N-1), p_{N-1}, c(N-1) \rangle \cdot \\ & \cdot | sp1', p'_1, c1' \rangle | sp2', p'_2, c2' \rangle \cdots | sp(P-1)', p'_{P-1}, c(P-1)' \rangle \end{aligned} \quad (30)$$

$$\begin{aligned} = & \sum_{\vec{m}, spN, spP'} A_N^s | spN, p_N, \vec{m} \rangle Prop_{\underline{a}; spN, spP'}(p_N) \\ & A_P^s(\text{with } p'_P \text{ outgoing}) | spP', p'_P, \vec{m} \rangle \end{aligned} \quad (31)$$

$$= \sum_{\vec{m}, spN, spP'} A_N^s | \vec{m} \rangle Prop_{\underline{a}; spN, spP'}(p_N) \sqrt{n} A_P^s | -\vec{m} \rangle \quad (32)$$

$$\approx \sum_{\vec{m}, \vec{m}', spN, spP'} A_N^s | \vec{m} \rangle Prop_{\underline{a}; spN, spP'}(p_N) A_P^s | \vec{m}' \rangle \quad (33)$$

The $|\vec{m}\rangle$ and $|\vec{m}'\rangle$ symbols denote that the N th, respectively P th, one of the external particles on the part of amplitude symbol $A_N^s |\vec{m}\rangle$, respectively $A_P^s |\vec{m}'\rangle$, is a particle with colour component \vec{m} , respectively \vec{m}' , while the irreducible representation of the colour degree of freedom is given as a consequence of the s -index. The first step, Eq. (31), consists in artificially writing the contribution A_{N+P-2}^s from a given singlet s as products of amplitudes A_N^s and A_P^s , with a completeness sum over all colour states. We use an orthonormal basis $|\vec{m}\rangle$, namely $\sum_{\vec{m}} |\vec{m}\rangle \langle \vec{m}| = 1$. This is consistent with the order of unity flavour normalisation, Eq. (14), of the amplitudes A_{N+P-2}^s , A_N^s and A_P^s .

As a further elaboration, we introduce an artificial double summation for the flavour (species) indices spN and spP' over $\#$ of species values. The statistical increase by a $\sqrt{\#}$ of species-factor for each of the two summations is compensated by inserting the propagator, Eq. (26). This is just repeating the result, obtained in subsection 6.1, that the number of species is irrelevant for splitting an amplitude A_{N+P-2} into amplitudes A_N and A_P connected by a propagator.

We remark that the amplitude denoted $A_P^s(\text{with } p'_P \text{ outgoing})|spP', p'_P, \vec{m} >$ in Eq. (31) is not normalised according to the same convention as the “order unity” amplitude obtained by just crossing our amplitude A_P^s . This is because we do not normalise the state with outgoing p'_P according to our “order unity” rule. We rather use the relation

$$< \underline{1}, \vec{0} | \underline{a}, \vec{m}; \underline{a}, -\vec{m} > \approx \frac{O(1)}{\sqrt{\dim(\underline{a})}}. \quad (34)$$

to obtain:

$$A_P^s(\text{with } p'_P \text{ outgoing})|spP', p'_P, \vec{m} > \approx \sqrt{\dim(\underline{a})} A_P^s | -\vec{m} > \quad (35)$$

The reason for this normalisation is that $A_P^s(\text{with } p'_P \text{ outgoing})|spP', p'_P, \vec{m} >$ then appears as a subpart of the Clebsch-Gordan construction of the A_{N+P-2} amplitude in the simplest way, i.e. without any extra $\sqrt{\dim(\underline{a})}$ factors. We are really just using Eq. (34) to convert the notation from outgoing to ingoing. So, in Eq. (32), it is meant that both the $|\vec{m} >$ and $| -\vec{m} >$ states are counted as ingoing. To get to the result Eq. (33), we use the following rule—which we often use in these random order one treatments: we take it that the sum of a series of statistically roughly independent terms of the same size equals a typical term times the square root of the number of terms.

From Eq. (33) we get by summation over the different contributions, i.e. by summation over s , that

$$A_{N+P-2} = \sum_{\underline{a}, \vec{m}, \vec{m}', spN, spP'} Prop_{\underline{a}, spN, spP'}(p_N) A_N | \vec{m} > A_P | \vec{m}' >. \quad (36)$$

But this is just the self-consistency equation we wanted to test! Therefore the order of magnitude unity, as we defined it for the $(N+P-2)$ -point function, coincides with what you get by composing an N -point function with a P -point function, that are each of order unity in our sense, with an intermediate propagator (that is of course summed over) . This fact we take as very suggestive for our chosen definition to be a good one. It means that it passed the ‘self-consistency check’, the criterion we gave at the beginning of section 6.

6.3 Extraction of definite colour_i (i=1,2,3) states

For the applications of the order unity amplitudes, it is not so nice to only have them defined for “random” colour states. We think of states with respect to

the three different $SU(3)$ groups in our AGUT-model, when we here talk about colour states. However, for simplicity, in this section we write formulae for only one of the “generation corresponding” colours. It would of course be useful to extract expressions for amplitudes of order one for states with definite colour indices.

As an example we shall take the cases of amplitude contributions that can occur in the large N_c limit, where we cannot use the ϵ -symbol in colours because it has (infinitely) many indices and only a finite number of external particles. Really let us, for our example here, imagine that to the singlet-contributions correspond amplitudes that can be simply written by means of colour kronecker deltas in the fundamental representations (= triplets for $SU(3)$) $\delta_{\alpha\beta}$, where then α and β run over the N_c colours. That is to say we take the contribution from one of the $\langle \text{singlet} |$ states to be proportional to a product of such kronecker deltas $\delta_{\alpha\beta}$

$$\begin{aligned} & \langle \text{singlet} | sp1, c1 \rangle | sp2, c2 \rangle \cdots | spN, cN \rangle \\ &= \text{“normalization”} \left(\prod_i \delta_{\alpha_i \beta_i} \right) \cdot \text{product of the } c1^{\alpha_i} \text{ etc.} \end{aligned} \quad (37)$$

where there can then be various numbers of indices attached to the various external states depending on their representations. Here the “normalization” factor is to be calculated so as to get the random states amplitude to come out right. If we only care for getting the large N_c factors right, we would not need to care for whether we symmetrise states with a few indices. Furthermore, in the cases of interest in our model, we have for one generation of colour all the time only triplets, singlets or antitriplets. So let us, as the example, consider that we only have contractions of triplets between the various external particles.

With $\langle \text{singlet} | \text{singlet} \rangle = 1$ the statistical average of the square modulus of the expression in Eq. (37) will be

$$\text{Average}(| \langle \text{singlet} | sp1, c1 \rangle | sp2, c2 \rangle \cdots | spN, cN \rangle |^2) = \frac{1}{n_1 n_2 \cdots n_N} \quad (38)$$

where n_i is the dimension of the representation (of colour) to which the random state $| spi, ci \rangle$ belongs. Or in other words

$$\langle \text{singlet} | sp1, c1 \rangle | sp2, c2 \rangle \cdots | spN, cN \rangle \approx \frac{1}{\sqrt{n_1 n_2 \cdots n_N}}. \quad (39)$$

Now putting the random states (in colour space) into the right-hand side of Eq. (37) leads to the result

$$\begin{aligned} & \text{“normalization”} \left(\prod_i \delta_{\alpha_i \beta_i} \right) \cdot \text{product of the } c1^{\alpha_i} \text{ etc.} \\ & \approx \text{“normalization”} \frac{1}{\sqrt[4]{n_1 n_2 \cdots n_N}} \end{aligned} \quad (40)$$

in the case we considered. We here used that two random triplets/ N_c -plets, $c2^\alpha$ and $c3^\beta$ say, have components of the order of $\frac{1}{\sqrt{N_c}}$ and the rule for summation of equal and independent terms giving a factor of the square root of their number. For each $\delta_{\alpha\beta}$ there are only N_c non-zero terms to sum and thus only one factor $\sqrt{N_c}$. Hence we have, for example

$$c2^\alpha \delta_{\alpha\beta} c3^\beta \approx \frac{1}{\sqrt{N_c}} = \frac{1}{\sqrt[4]{n_2 n_3}}. \quad (41)$$

So we conclude that, for the consistency of Eq. (37), we must take

$$\text{“normalization”} \approx \frac{1}{\sqrt[4]{n_1 n_2 \cdots n_N}} \quad (42)$$

without any further N_c factors.

Now the full “of order unity” amplitude was defined in Eq. (20) as the sum over all the possible singlets. In cases where we only have singlets that can be written as simple kronecker delta contractions, as in the “large N_c ” case we treated here, it follows that the full amplitude is obtained by summing up—one for each singlet—the expressions

$$\begin{aligned} & \langle \text{singlet} | sp1, c1 \rangle | sp2, c2 \rangle \cdots | spN, cN \rangle \\ &= \frac{\text{“O(1)-factor”}}{\sqrt[4]{n_1 n_2 \cdots n_N}} \left(\prod_i \delta_{\alpha_i \beta_i} \right) \cdot \text{product of the } c1^{\alpha_i} \text{ etc.} \end{aligned} \quad (43)$$

to get

$$\begin{aligned} & A_N(sp1, c1; sp2, c2; \cdots; spN, cN) \\ &= \sum_{\text{the singlets}} \frac{\text{“O(1)-factor”}}{\sqrt[4]{n_1 n_2 \cdots n_N}} \left(\prod_i \delta_{\alpha_i \beta_i} \right) \cdot \text{product of the } c1^{\alpha_i} \text{ etc.} \end{aligned} \quad (44)$$

The fourth root normalisation factor could be distributed through Eq. (43) or (44) so as to give a factor $\frac{1}{\sqrt{N_c}}$ following each kronecker delta. In other words if, instead of the kronecker deltas above, we use the combination

$$\frac{\delta_{\alpha\beta}}{\sqrt{N_c}} \quad (45)$$

where N_c is the number of colours (for the $SU(3)$ -group of the generation in question, of course $N_c = 3$), we do not need the fourth root denominators. That is to say we can write e.g. Eq. (43) as

$$\begin{aligned} & \langle \text{singlet} | sp1, c1 \rangle | sp2, c2 \rangle \cdots | spN, cN \rangle \\ &= \text{“O(1)-factor”} \left(\prod_i \frac{\delta_{\alpha_i \beta_i}}{\sqrt{N_c}} \right) \cdot \text{product of the } c1^{\alpha_i} \text{ etc.} \end{aligned} \quad (46)$$

6.4 Implications of our order unity choice

Now that we have chosen a prescription for making the order unity concept more precise, what does this choice imply for the possibility of finding large N_c factors in our mass predictions?

As an example we consider the characteristic prediction of our model: that if the diagonal elements dominate the mass matrices, then the quarks and charged lepton in the same generation have order of magnitudewise the same masses. Could this prediction be changed by, say, an N_c factor?

In the model we use we have assumed that there are several Dirac fermions, as well as other particles, with masses of the order of the Planck scale for all quantum number combinations we can think of. So, in particular, the quantum numbers of say a right-handed b-quark occur on several particles which are very heavy. This should be understood in the terminology of Weyl particles as follows: There are several right-handed Weyl fields with the quantum number combination of the right-handed b-quark and in addition several, but one less, left-handed Weyl fields with this quantum number combination. So just one linear combination of these right-handed Weyl fermions is left without a partner and is thus massless compared to the Planck scale. It only gets its mass, at the end, by being paired with a Weyl field having the quantum numbers of the left-handed b-quark and, only then, under the influence of the Weinberg-Salam Higgs field and other Higgs fields in our model. It thus gets a mass much smaller than the Planck scale and this particle is the b-quark.

We imagine now that we have integrated out the fields of mass of the order of the Planck scale and replaced their effects by effective N-point amplitudes for the lighter particles. For example the effective amplitude $A_{2+h}^{(mass)}(l, r, W, T, \dots)$ that describes the scattering amplitude between the various particles described by the symbols l, r, W, T, \dots which stand for their names and states: the l and the r are the left- and right-handed quarks or leptons, and the W, T, \dots stand for our various Higgs fields. We shall be especially interested in the case of the states of the Higgs fields W, T, \dots being those superpositions in which they occur in the vacuum. But we just wrote “ W, T, \dots ” as an example and we can, of course, write a similar expression with any combination of our Higgs fields. Further h is the number of the Higgs fields W, T , etc. and the upper index “ $(mass)$ ” means that we ignore the kinetic part, or equivalently that we take the fermion external momenta to zero.

If the external states W, T, \dots are thought of as the vacuum condensate states—having of course zero four momentum—the $A_{2+h}^{(mass)}(l, r, W, T, \dots)$ -amplitude becomes the two point function corresponding to the effective mass term in the Lagrangian for the Dirac particle formed from the Weyl particles l and r .

If the kinetic energy term in the Lagrangian formulation were simply \not{p} times a unit matrix, we could see that this “two”-point amplitude $A_{2+h}^{(mass)}$ is the mass-matrix (in colour space), but now the kinetic term coefficients are also random corresponding to being of order unity in the sense which we defined. We could

renormalize the fields so as to bring the kinetic term to the standard form or, equivalently, we could divide by the ratio of the kinetic term to \not{p} . Since the propagators $prop_R$ and $prop_L$ are defined as inverses of the kinetic terms, we really have to correct the mass-amplitude $A_{2+h}^{(mass)}$ by a factor $\frac{prop}{\not{p}}$. We now consider the mass squared m^2 for the Dirac particle formed, which can be gotten from the relation

$$\begin{aligned} Tr \left[A_{2+h}^{(mass)}(l, r, W, T, \dots) \frac{prop_R}{\not{p}} \bar{A}_{2+h}^{(mass)}(r, l, W, T, \dots) \frac{prop_L}{\not{p}} \right] \\ = dim(\text{representation}) m^2. \end{aligned} \quad (47)$$

Here these propagators are really just matrices approximately proportional to the identity $\frac{prop_R}{\not{p}} \approx \sqrt{n} \delta_{\vec{m}, \vec{m}'}$ in colour space, where n is the number of colour states for the representation of l or r (assumed equal), $n = dim(\text{representation})$. We assume that l and r belong to the same representation with respect to colour in order that the mass term shall not violate colour conservation. The trace, Tr , stands for the trace in the space of various—really colour—states in the l and r channels. If, by colour symmetry, they all have the same mass (squared) we just get the dimension of the representation of the l or r state spaces from this trace-summation, explaining the $dim(\text{representation})$ factor on the right-hand side of Eq. (47).

From our prescription Eq. (20) for “order unity” amplitudes, we have

$$\begin{aligned} A_{2+h}(r; l; W; \dots; T) = \\ \sum_{\text{singlet}_N} \text{“O(1) – factor”} < \text{singlet}_N || r > | l > | W > \dots | T >. \end{aligned} \quad (48)$$

provided the external states are taken as “random” states—random colour superpositions first of all. Now, however, we want to consider definite colour states and we can make use of the estimate, Eq. (46), in subsection 6.3 to obtain:

$$\begin{aligned} A_{2+h}(r; l; W; \dots; T) \\ = \sum_{\text{the singlets}} \text{“O(1)-factor”} \left(\prod_i \frac{\delta_{\alpha_i \beta_i}}{\sqrt{N_c}} \right) \cdot \text{product of the } c1^{\alpha_i} \text{etc.} \end{aligned} \quad (49)$$

We have to imagine that, for the various Higgs fields W , T , etc., we have some colour contraction kronecker deltas *between different generation colours*. We note that all the Higgs fields in our model must conserve the diagonal colour group, which is identified with the QCD-colour group. So the typical Higgs field, like T in our model, must take expectation values which are invariant under this diagonal $SU(3)$, although they break spontaneously the separate $SU(3)$ ’s of the various generations. That is to say that, for example, T could be triplet and anti-triplet under the 2nd and 3rd generation $SU(3)$ ’s respectively, and

thus its vacuum expectation value would be proportional to an inter-generation kronecker delta, having one index of generation 2 and the other of generation 3.

In the simple case where we can neglect contributions containing ϵ -symbols—the large N_c approximation—we would simply get several kronecker deltas contracted in circular chains, because at the end there should be no free indices in a colourless expression such as Eq. (47). Each ring of circularly contracted kronecker deltas is quickly reduced to just the trace of one kronecker delta and thus to N_c , the number of colours, which is the same for all the generation colour groups. The dominant term will be the one corresponding to that $|singlet\rangle$ which leads to the highest number of circular chains, because that will give the highest number of N_c -factors.

For example for a diagonal mass matrix element, in which we have the same generation of colour index on the l and the r states, it will pay best to get a kronecker delta directly contract l and r ; otherwise potential further contraction loops would miss the chance of existing, because the Higgs fields put between the l and the r in the contraction chains would already have been used. We can use this observation to argue that the trace on the left-hand side of Eq. (47) for a quark of a certain generation will be just bigger, by having one contraction-loop more, than the corresponding lepton in the same generation. That means then that this trace will be N_c times bigger for a quark than for a lepton in the same generation—if their masses are given by the diagonal mass matrix elements (i.e. except for c and t in our model). From the presence of the factor $dim(\text{representation})$ in Eq. (47), it then follows that to this “large N_c approximation”, when diagonal matrix element dominate, the quark and charged lepton masses in the same generation are equally big (up to order unity factors which do not contain N_c factors, as one could perhaps have feared.).

We have thus seen that one of the major predictions of our model—the order of magnitude degeneracy of masses in the same generation as long as they are dominated by diagonal matrix elements—is NOT modified by any big factor even if the colour number, $N_c = 3$, should be considered “bigger than of order unity”. It follows that one can expect a higher accuracy than having to ignore a deviation by a factor $N_c = 3$ in our predictions. Indeed our fits are accurate to better than to a factor 3. We only showed this for the diagonal mass matrix elements and with ignored ϵ ’s, but it is not difficult to extend the argument to include the possibility of ϵ ’s for the intrageneration mass ratios.

It is somewhat complicated to calculate the colour-counting corrections for other matrix elements, but preliminary studies suggest that even for the off-diagonal elements the corrections to our fit are not very big.

7 The expected uncertainty

According to our discussion of the many different permutations of the order of the Higgs field tadpoles in the Feynman diagrams contributing to a given mass

matrix element, it could easily turn out that the mass matrix elements appear in our model as sums over rather many terms, all having in principle the same order of magnitude. There is also the possibility of having several fundamental scale particles with the same quantum numbers. If indeed there are many terms of the same approximate size, adding up with some random order of unity factors, we should be able to apply the central limit theorem to conclude that the sum, considered as a stochastic variable z , will have a Gaussian distribution in the complex plane (really a two-dimensional Gaussian distribution).

But now we mainly care for the order of magnitudes of the mass matrix elements and calculate the χ^2 -like quantity for the logarithm of the variable rather than for the variable itself. So we shall now consider what the spread in the logarithm is for a variable that is Gaussian distributed. We shall assume, what because of their assumed random phases is true for the mass matrix elements in our model, that the average is zero. Then, for dimensional reasons, there is no way in which this fluctuation can depend on the width of the Gaussian distribution of the quantity itself.

We now calculate the root mean square fluctuation of the logarithm of the stochastic variable z :

$$\langle (\log z)^2 \rangle - \langle \log z \rangle^2 \quad (51)$$

$$= \frac{\int \exp(-|z|^2) (\log |z|)^2 d^2 z}{\int \exp(-|z|^2)} - \left(\frac{\int \exp(-|z|^2) \log |z| d^2 z}{\int \exp(-|z|^2) d^2 z} \right)^2 \quad (52)$$

$$= \frac{\int_0^\infty \exp(-t) (1/2 \log t)^2 dt}{\int_0^\infty \exp(-t) dt} - \left(\frac{\int_0^\infty \exp(-t) 1/2 \log t dt}{\int_0^\infty \exp(-t) dt} \right)^2 \quad (53)$$

$$= \frac{\Gamma''(1)}{4\Gamma(1)} - \left(\frac{\Gamma'(1)}{2\Gamma(1)} \right)^2 = \frac{\psi'(1)}{4} = 0.4112 \quad (54)$$

If all the masses in nature really followed our model in a statistical way they would deviate in the logarithm from the predicted values statistically by a factor of $\sqrt{\psi'(1)/4} = 0.6412$. Now we make a fit with $12 - 4 = 8$ degrees of freedom, and so we expect the pseudo-chisquared of Eq. (13) to be $8 \times \psi'(1)/4 = 8 \times 0.4112 = 3.29$. This to be expected pseudo-chisquared is of a very similar order of magnitude to the pseudo-chisquareds which we find in our fits. So roughly our agreement is similar to the expected one. But actually we got a somewhat better agreement than one should have expected for our presented fits!

We should bear in mind, however, that the fluctuation of the pseudo-chisquared itself is of the order of 35%, so that the predicted pseudo-chisquared of 3.3 ± 1.2 should be counted as only one standard deviation off if it were measured to be 2.1. It is, of course, clear that if we fit 81 models and look at the ones with the smallest pseudo-chisquared, we should get some that have an anomalously low pseudo-chisquared. However, we investigated the distribution of our original 81 approximate fits and found a pseudo-chisquared fluctuating around 2.1 with a spread of 0.9. This means that the major part of these fits work too well! A possible explanation for this is that all these 81 fits are not really very different

and should be considered roughly as just one fit, accidentally fitting too well by one standard deviation. An alternative explanation could be that some of our predictions are actually valid more accurately than in our model due, say, to some physics which our model has missed. For example we have the order of magnitude—but not exact— $SU(5)$ mass predictions in our model. It could easily be that an exact $SU(5)$ could lead to too good fits for our model.

8 Conclusion

We have improved our previous AGUT model fits [1] to the quark-lepton mass spectrum in a couple of ways:

First we have taken into account the number of different permutations of the Higgs field attachments to the Feynman diagram contributing to a given mass matrix element. This is a necessary correction that should be expected to be there. Secondly, in order to readjust the fit, it seemed necessary to let the VEV of the Higgs field S , which was previously set equal to unity, also be a fitted parameter. Because of the S field now being also relevant, there are many variants of the model obtained by different choices of the gauge quantum numbers of the Higgs fields W , T , etc., which were previously only determined modulo those of S . This gives a lot of models, but they are in reality not very different.

We proposed a more careful definition of what an order one coupling assumption shall mean and investigated whether it could give rise to correction factors of order $N_c = 3$. Such large corrections do not in fact appear, at least in our prediction of intrageneration mass degeneracy (except for the top and charm quark masses, which are dominated by off-diagonal mass matrix elements). Thus there appears to be no reason why we should not trust our naive calculations to an accuracy better than a factor of 3.

So, as far as the charged quark and lepton mass matrices (neutrinos are a problem for several reasons [10]) are concerned, we managed to successfully fit the 9 masses, 3 mixing angles, and the CP-violation strength with our AGUT-model, i.e. with the gauge group $SMG^3 \times U(1)_f$, and five Higgs fields W , T , ξ , S and ϕ_{WS} . In addition to the usual Weinberg Salam Higgs vacuum expectation value—which in our model is replaced by ϕ_{WS} —we only used the vacuum expectation values of the four other Higgs fields as free parameters in our fits. It should then be remarked though that:

1) With the slightly complicated gauge group representations of our Higgs fields, some fitting of discrete quantum numbers could be said to have taken place.

2) The expectation value of the Higgs field S is so close to unity that we only barely need it in the fits, and so we should hardly count its value as a parameter. Its value is seldom fit to be less than 0.3 and it never needs to be more than 1.

3) Really we have a significant amount of ambiguity in the discrete quantum numbers mentioned under 2) in the sense that many proposals, deviating from each other by adding the quantum numbers of the field S to one or several of the other Higgs fields, give very good fits.

The fits are so good that many of the fitting proposals mentioned under point 3) even fit the data slightly better than could be expected. We calculated an expectation for the χ^2 , defined in Eq. (13) for the logarithmic comparison with data, from the hypothesis that the mass matrix elements at first have Gaussian distributions. This expected value for χ^2 is 3.3 ± 1.2 , which roughly agrees with the χ^2 values of our fits. For a lot of the 81 models considered, we actually obtained a somewhat smaller χ^2 than this theoretically predicted one. So we even have reasonably good agreement concerning the degree of accuracy of our fit and must conclude that the model fits the charged fermion mass matrices at least as well as one should expect.

9 Acknowledgements

This work was supported by EU funds CHRX-CT-94-0621, INTAS 93-3316, INTAS RFBR-95-0567, SCI-0430-C (TSTS). We should also like to thank the Slovenian Ministry of Science for supporting the Bled workshops, where some of this work was developed.

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